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Topology and bifurcations of the invariant level sets of a Fokker–Planck Hamiltonian through two coupled anisotropic quartic anharmonic oscillators

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Abstract

Integrable Hamiltonians with velocity-dependent potentials, including those of Fokker–Planck Hamiltonians $H = 1/2(p_x^2 + p_y^2) + k_x p_x + k_y p_y$, are constructed from integrable Hamiltonians of type $H = 1/2(p_x^2 + p_y^2) + V(x, y)$. In order to carry out the analytical investigations, we convert the problem into that of two coupled anisotropic quartic anharmonic oscillators using certain canonical transformations; afterwards we give a complete description of the real phase space topology of the system. We give also an explicit periodic solution for singular common-level sets of the first integrals. All generic bifurcations of Liouville tori were determined analytically and numerically.

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1. Introduction

Integrability is clearly a central issue in understanding the origins and implications of the behaviour of dynamical systems. Physically interesting integrable systems are rare, and consequently it stirs up considerable excitement when one is discovered. Moreover, until now, no systematic procedure has been established for the identification of an integrable system.

The question of integrability of a dynamical system was raised soon after Newton formulated the equations of motion of three bodies in a gravitational field. By integrability (often referred to as complete integrability) of a Hamiltonian system with N degrees of freedom we mean the existence of N-analytic, single-valued, global integrals of the motion which are functionally independent and in involution. When this is the case the Liouville theorem entails that the problem can be solved by quadratures.

The concepts of integrability have been applied to an increasing number of physical systems; among others the integrability of the Fokker–Planck equation has received particular attention [1, 2]. Many statistical systems can be described by a Markov process whose probability density P is a solution to the Fokker–Planck equation. In the weak noise limit one can make a semiclassical-type approximation for P and in that limit the equation for the stationary probability density reduces to the form

$$\frac{1}{2}Q_{ij}\frac{\partial\Phi}{\partial q_i}\frac{\partial\Phi}{\partial q_j}+K_i\frac{\partial\Phi}{\partial q_i}=0 \tag{1.1}$$

where Φ is the non-equilibrium potential (action) [1]. When studying the question of solvability of equation (1.1) a mechanical analogy turns out to be helpful. It should be noted that the nonlinear equation (1.1) has the form of a Hamilton–Jacobi equation

$$H\left[q,\frac{\partial S}{\partial q}\right]=E \tag{1.2}$$

where H denotes a Hamiltonian of a mechanical system with generalized coordinates q_i, E and $S(q)$ denote the energy and the action, respectively, and $\partial_i S=\partial_i\Phi=p_i$ defines the momenta. A comparison with equation (1.1) gives $E=0$ and

$$H(q,p)=\tfrac{1}{2}Q_{ij}p_ip_j+K_i(q)p_i \tag{1.3}$$

as the Hamiltonian associated with the macroscopic system. Since H can be uniquely constructed from the coefficients of the Fokker–Planck equation we call (1.3) a Fokker–Planck Hamiltonian whose first term can be interpreted as a kinetic energy term with an anisotropic mass tensor. The second one, however, is different from the usual potential energy. Similar terms appear when describing the motion of charged particles in an external magnetic field, $K_i(q)$ is, thus, analogous to a vector potential.

The purpose of the present paper is to bring out the connection via a canonical transformation existing between the Fokker–Planck Hamiltonian (1.3) and the perturbed two-dimensional anisotropic oscillator, and give a detailed description of the real phase space topology of the system. Hence in what follows we restrict our study to a system with two variables (four-dimensional Hamiltonian phase space q_1, q_2, p_1, p_2) and with a constant diagonal anisotropic mass tensor, described by a Fokker–Planck Hamiltonian of the form

$$\begin{aligned}H(p,q,a,b,c)=&-\frac{1}{3a^2}p_1^2+\frac{4}{3b^2}p_2^2+\left(-\frac{c}{a}+2aq_1^2-\frac{5b^2}{a}q_2^2\right)p_1\\&+\left(\frac{2c}{b}+7bq_2^2+4aq_1q_2\right)p_2\end{aligned} \tag{1.4}$$

where a, b and c are constants.

The main difficulty one can encounter when studying a Hamiltonian of this type, in addition to that of integrability, is the separability of its variables (p and q). However, with the help of the following canonical transformation:

$$\begin{aligned}q_1&=\frac{1}{3a}\left(\frac{p_y}{y}-\frac{5}{4}\frac{p_x}{x}\right) \qquad & p_1&=a(x^2+y^2)\\ q_2&=\frac{1}{3b}\left(\frac{p_x}{x}-\frac{p_y}{y}\right) \qquad & p_2&=b\left(x^2+\tfrac{5}{2}y^2\right)\end{aligned} \tag{1.5}$$

we have reduced (1.4) to the well known separable form describing two coupled anisotropic quartic anharmonic oscillators whose Hamiltonian flow is generated by the Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+c(x^2+4y^2)+x^4+6x^2y^2+8y^4 \tag{1.6}$$

which is the integrable extension case of the quartic potential [3].

Then the equations for the vector field X_H, defined by $\omega(X_H) - \mathrm{d}H$ are given by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= p_x \qquad \frac{\mathrm{d}p_x}{\mathrm{d}t} = -4x^3 - 12xy^2 - 2cx \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= p_y \qquad \frac{\mathrm{d}p_y}{\mathrm{d}t} = -32y^3 - 12x^2y - 8cy \end{aligned} \tag{1.7}$$

on the canonically symplectic phase space $M^4_{\Re}$ with the symplectic structure

$$\omega^{(2)} = \sum_{j=1}^{2} \mathrm{d}p_j \wedge \mathrm{d}q_j.$$

As is well known, there exists the second invariant [3] commuting with (1.6)

$$\begin{aligned} G = {} & p_x^4 + 4x^2(c + x^2 + 6y^2)p_x^2 - 16x^3yp_xp_y + 4x^4p_y^2 \\ & +4x^4(c^2 + 2cx^2 + 4cy^2 + x^4 + 4x^2y^2 + 4y^4). \end{aligned} \tag{1.8}$$

According to the classical Liouville theorem, for non-critical values of H and G the generic invariant manifolds $M^4_{\Re}$ (i.e. the generic level sets) of a completely integrable Hamiltonian system consist of tori or cylinders. In order to describe the topological nature of the Hamiltonian flows on the whole phase space, i.e. the topology of the real level sets $M^4_{\Re}$

$$M^4_{\Re}(h, g) = \left\{(X, P) \in \Re^4 : H = h, G = g\right\}$$

where h and g are the values of the first integrals H and G; we have to determine all generic bifurcations of Liouville tori which correspond to the topological nature of the non-generic invariant manifolds, and then explain how the invariant manifolds topologically fit together, as the values of the constants of motion vary. At last we give an explicit periodic solution for non-generic invariant manifolds $M^4_{\Re}$ and a numerical illustration of the bifurcations studied above.

2. Topological analysis

The Hamilton–Jacobi equation corresponding to the system (1.6) separates into u, v, p_u, p_v coordinates defined in [3–5]

$$u = \frac{p_x^2 + \sqrt{g}}{x^2} + 2x^2 + 4y^2$$

$$v = \frac{p_x^2 - \sqrt{g}}{x^2} + 2x^2 + 4y^2$$

$$p_u = \frac{xp_xp_y - yp_x^2 + 4x^2y^3 + y(2x^4 - \sqrt{g})}{8x(xp_y - 2yp_x)}$$

$$p_v = \frac{xp_xp_y - yp_x^2 + 4x^2y^3 + y(2x^4 + \sqrt{g})}{8x(xp_y - 2yp_x)}.$$

It is easy to check that (x, y, p_x, p_y) can be expressed in terms of the u, v coordinates in the following way:

$$\begin{aligned}
x^2 &= \frac{2\sqrt{g}}{u-v} \\
xp_x &= \sqrt{g}\frac{\sqrt{Z_2(v)}-\sqrt{Z_1(u)}}{(u-v)^2} \\
y^2 &= \frac{u+v}{8} - \frac{\sqrt{g}}{u-v} - \frac{\left(\sqrt{Z_1(u)}-\sqrt{Z_2(v)}\right)^2}{16(u-v)^2} \\
yp_y &= \frac{\sqrt{Z_1(u)}+\sqrt{Z_2(v)}}{16} + \frac{p_x}{2x}\left(x^2+6y^2+c+\frac{p_x^2}{2x^2}\right).
\end{aligned} \tag{2.1}$$

In the above expression $Z_1(u)$ and $Z_2(v)$ denote the polynomials

$$\begin{aligned}
Z_1(u) &= 2u^3+4cu^2-8u(c^2+2h+\sqrt{g})-16c(c^2+2h+\sqrt{g}) \\
Z_2(v) &= 2v^3+4cv^2-8v(c^2+2h-\sqrt{g})-16c(c^2+2h-\sqrt{g}).
\end{aligned} \tag{2.2}$$

Differentiating (2.1) with respect to the variable time and using the system (1.7), we obtain the following expressions:

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\mathrm{d}v}{\mathrm{d}t} &= \sqrt{Z_1(u)}+\sqrt{Z_2(v)} \\
\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{\mathrm{d}v}{\mathrm{d}t} &= \sqrt{Z_1(u)}-\sqrt{Z_2(v)}.
\end{aligned} \tag{2.3}$$

The differential equations satisfied by u, v are

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= \sqrt{Z_1(u)} \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= \sqrt{Z_2(v)}.
\end{aligned} \tag{2.4}$$

Thus u, v (hence x, y, p_x, p_y and afterwards q_1, q_2, p_1, p_2) can be expressed in terms of Weierstrass elliptic functions.

2.1. Topology of generic invariant manifolds

In this section we shall describe the topological type of $M^4_{\Re}$ for all generic constants $g, h \in \Re$.

Remark. The Hamiltonian flow (1.7) is considered as a complex system; through equation (2.1) one notices that variables $x, y, p_x, p_y \in \mathrm{i}\Re$, but h and g are real constants.

In order to give a complete description of the topology of $M^4_{\Re}$, we find first the bifurcation diagram B, i.e. the set of the critical values of the energy–momentum mapping

$$(u, v, p_u, p_v) \to (H, G)$$

It turns out (such as in Hénon–Heils [6], Goryatchev–Tchaplygin top [7–9], Kolossoff potential [10, 11] and the swinging Attwood machine [12]) that B is exactly the discriminant locus of the polynomials $Z_1(u)$ and $Z_2(v)$ whose coefficients are functions in h and g,

$$B = B_1 \cup B_2$$

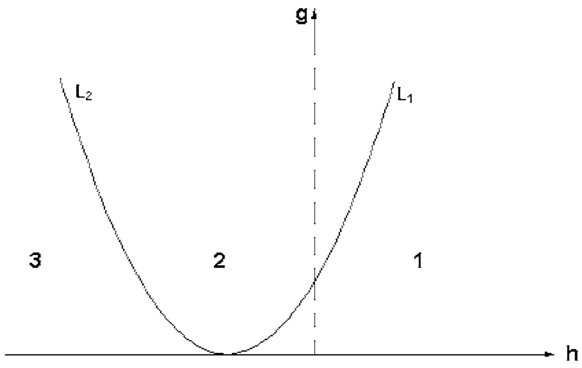

Figure 1. Diagram of bifurcation $B \cap \{c = \text{constant}\}$

Table 1. Real roots of the polynomials $Z_1(u)$ and $Z_2(v)$ for $(h, g, c) \in M^4_{\Re} \setminus B$.

Domain	Real roots of u and v	
1	$u_1 < u_2 < u_3$	$v_1 < v_2 < v_3$
2	$u_1 < u_2 < u_3$	v_1
3	u_1	v_1

where

$$B_1 = \{(h, g, c) \in \Re^3 / \operatorname{disc}(Z_1(u)) = 0\}$$

$$B_2 = \{(h, g, c) \in \Re^3 / \operatorname{disc}(Z_2(v)) = 0\}.$$

It is clear that the topological type of $M^4_{\Re}$ may change only as (h, g, c) passes through B and that in each connected component of the sets $\Re^3 \setminus B$, the level set $M^4_{\Re}$ has the same topological type. Note that the bifurcation set $B \subset \Re^3\{h, g, c\}$ is invariant under the map

$$(h, g, c > 0) \to (h, g, c < 0)$$

and the topological type of the level set $M^4_{\Re}$ is one and the same at the points $(h, g, c > 0)$ and $(h, g, c < 0)$, thus it is enough to consider $c > 0$.

Definition. *The sets $B \cap \{c = c_1\}$ and $B \cap \{c = c_2\}$ are topologically equivalent so there exist continuous functions $c = c(s)$, $s \in [0, 1]$, such that $c(0) = c_1$, $c(1) = c_2$, and all sets $B \cap \{c = c(s)\}$, $s \in [0, 1]$ are homeomorphic each to other.*

Theorem. *The set $\{\Re^3 \setminus B\} \cap \{c > 0\}$ consists of three connected domains which do not intersect with each other, as shown in figure 1. The topological type of $M^4_{\Re}$ is a disjoint union of two-dimensional tori, two-dimensional cylinders and real plane $\Re^2$ as shown in table 2.*

Proof. Consider the complexified system

$$M^4_{\not{C}} = \{(x, y, p_x, p_y) \in \not{C}^4 : H = h, G = g; u \neq v\}$$

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + c(x^2 + 4y^2) + x^4 + 6x^2y^2 + 8y^4$$

$$G = p_x^4 + 4x^2(c + x^2 + 6y^2)p_x^2 - 16x^3yp_xp_y + 4x^4p_y^2$$
$$+4x^4(c^2 + 2cx^2 + 4cy^2 + x^4 + 4x^2y^2 + 4y^4).$$

Consider also the elliptic curves

$$\Gamma_1 : \{w_1^2 = Z_1(u)\} \qquad \text{and} \qquad \Gamma_2 : \{w_2^2 = Z_2(v)\}$$

and the corresponding Riemann surfaces R_1 and R_2 of the same genus $g_1 = g_2 = 1$. Define the natural projection

$$\pi : M^4_{\not\subset} \to \Gamma_1 \otimes \Gamma_2$$

(where $\otimes$ is the symmetric product), and the complex conjugation on $M^4_{\not\subset}$:

$$\tau : (x, y, p_x, p_y) \to (\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y). \tag{2.5}$$

Consider also the natural projection η on the Riemann surface $R = R_1 \otimes R_2$ given in (u, v) coordinates by

$$\eta : (u, v) \to (\bar{u}, \bar{v}).$$

It induces an involution on $M^4_{\not\subset}$ by the natural projection π. Formulae (2.1) and (2.4) imply that this involution η coincides with the complex conjugation (2.5) on $M^4_{\not\subset}$. Hence in order to describe $M^4_{\Re} = \text{Re}(M^4_{\not\subset}) \cap \{c > 0\}$ it is sufficient to study the projection π. □

Definition. *A connected component of the set of fixed points of* τ *on the curves* Γ_1 *and* Γ_2 *is called an oval.*

To determine the ovals of Γ_1 and Γ_2, it suffices to study the real roots of the polynomials $Z_1(u)$ and $Z_2(v)$ for different values of h and g as shown in table 1. Using conditions $x \neq 0$, $y \neq 0$ from (1.5), (2.1), (2.4) and the condition that $(u, v, p_u, p_v) \in \Re^4$, then, we find exactly two admissible ovals whose projections on the u-plane and the v-plane are given by Δ_1 and Δ_2 (see table 2).

The product of the admissible ovals in $\Gamma_1 \otimes \Gamma_2$ and the projection π of $M^4_{\Re}$ such as $M^4_{\Re} = \pi^{-1}(\Gamma_1 \otimes \Gamma_2) = \Delta_1 \otimes \Delta_2$ gives:

- $M^4_{\Re}$ is a disjoint union of tori, two cylinders and the real plane $\Re^2$ in domain 1;
- $M^4_{\Re}$ is a disjoint union of a cylinder and the real plane $\Re^2$ in domain 2;
- $M^4_{\Re}$ is the real plane $\Re^2$ in domain 3.

T and C denote a two-dimensional torus and a two-dimensional cylinder, respectively.

Table 2. Admissible ovals and topological type of $M^4_{\Re}$ for $(h, g, c)M^4_{\Re} \setminus B$.

	Projection of the admissible ovals on		
Domain	u-plane Δ_1	v-plane Δ_2	Topological type of $M^4_{\Re}$
1	$[u_1, u_2] \cup [u_3, \infty[$	$[v_1, v_2] \cup [v_3, \infty[$	$T + 2C + \Re^2$
2	$[u_1, u_2] \cup [u_3, \infty[$	$[v_1, \infty[$	$C + \Re^2$
3	$[u_1, \infty[$	$[v_1, \infty[$	$\Re^2$

Table 3. Generic bifurcations of the level set $M^4_{\Re}$ passing from domain i to domain j.

$1 \to 2$	$2 \to 3$
$T + 2C + \Re^2 \to C + \Re^2$	$C + \Re^2 \to \Re^2$

2.2. Topology of non-generic invariant manifolds

Suppose now that the constants h and g, are changed in such a way, that (h, g) passes through the bifurcation diagram. Then, the topological type of $M^4_{\Re}$ may change and the bifurcation of $M^4_{\Re}$ takes place.

In this section, we study the description of all generic bifurcations of the topological type of $M^4_{\Re}$ (see table 3). The Fomenko classification of bifurcation of Liouville tori [13] cannot be applied to the two coupled anisotropic quartic anharmonic oscillators as its invariant level sets contain a non-compact component (cylinder and $\Re^2$ plane).

Thus, we can have the type of bifurcation in table 3, as shown in figure 1.

To prove that, it suffices to look at the bifurcation of roots of the polynomials $Z_1(u)$ and $Z_2(v)$ as shown in figure 2.

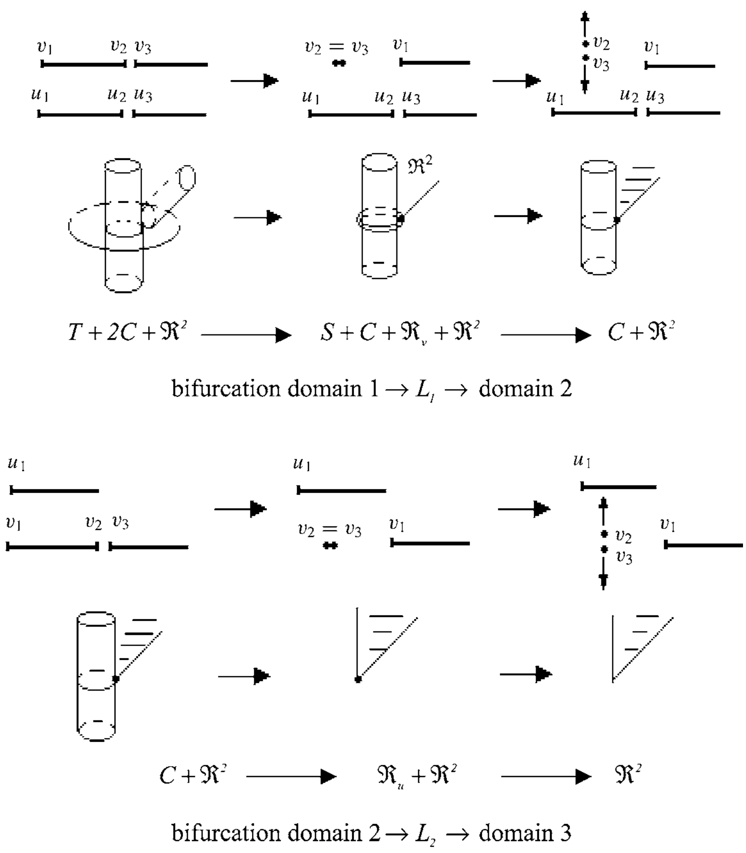

Figure 2. Correspondence between bifurcation of roots of polynomials $Z_1(u)$ and $Z_2(v)$ and bifurcation of $M^4_{\Re}$.

3. Periodic solutions

When the bifurcation of Liouville tori takes place, the level set $M^4_{\Re}$ becomes completely degenerate. Then we can have exceptional families of periodic solutions. It is seen from table 4 that, if (h, g) is on the smooth curve L_1 delimiting the domains 1 and 2 (see figure 1), then $M^4_{\Re}$ contains a unique isolated circle S which is a periodic solution.

Consider now a fixed periodic solution belonging to the curve L_1. The parameter u takes its values in the admissible intervals $[u_1, u_2] \cup [u_3, \infty[$ and $v_2 = v_3 = 0$ is equal to the double root of the polynomial $Z_2(v)$ (see table 4). The values of the first integrals H and G on the curve L_1 are linked by $g = (2h + c^2)^2$.

Then we obtain from equation (2.1) the following parametrization of the fixed periodic solution:

$$x^2 = \frac{2\sqrt{g}}{u}$$

$$xp_x = -\sqrt{g}\frac{\sqrt{Z_1(u)}}{(u-v)^2}$$

$$y^2 = \frac{u}{8} - \frac{\sqrt{g}}{u} - \frac{Z_1(u)}{16u^2}$$

$$yp_y = \frac{\sqrt{Z_1(u)}}{16} + \frac{p_x}{2x}\left(x^2 + 6y^2 + c + \frac{p_x^2}{2x^2}\right).$$

The differential equation satisfied by u is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sqrt{Z_1(u)}.$$

Thus $u = u(t)$ and hence $x(t)$, $y(t)$ can be expressed in terms of elliptic functions. The period of the solution $u(t)$ is

$$T = \oint \mathrm{d}t = \oint \frac{\mathrm{d}u}{\sqrt{Z_1(u)}} = 2\int_{u_1}^{u_2} \frac{\mathrm{d}u}{Z_1(u)}.$$

We obtain

$$T = \frac{2}{\sqrt{u_3 - u_1}}\,\mathrm{sn}^{-1}\left(1, \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}\right) = \frac{2}{\sqrt{u_3 - u_1}} K\left(\sqrt{\frac{u_2 - u_1}{u_3 - u_1}}\right)$$

where K is the complete elliptic integral of the first kind, and sn is the Jacobi elliptic function.

The roots of the polynomials $Z_1(u)$ on the curve L_1 are such that for $u_1 < u_2 < u_3$,

$$u_1 = -u_3 = -2\sqrt{2h + c^2}\sqrt{\mathrm{sign}(2h + c^2) + 1}$$

$$u_2 = -2c.$$

$u(t)$ is given by solving the Jacobi inversion problem [14]

$$t = \int_0^t \mathrm{d}t = \int_{u_1}^{u} \frac{\mathrm{d}u}{Z_1(u)}.$$

Table 4. Topological type of $M^4_{\Re}$ on diagram B.

Curves	Δ_1	Δ_2	$M^4_{\Re} = \Delta_1 \otimes \Delta_2$
L_1	$[u_1, u_2] \cup [u_3, \infty[$	$\{v_2 = v_3 = 0\} \cup [v_1, \infty[$	$S + C + \Re_v + \Re^2$
L_2	$[u_1, \infty[$	$\{v_2 = v_3\} \cup [v_1, \infty[$	$\Re_u + \Re^2$

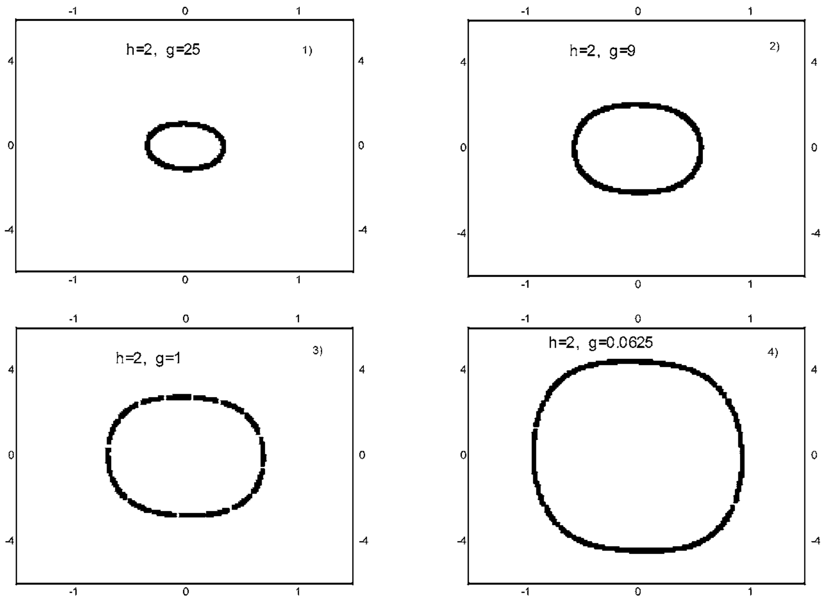

Figure 3. Surface-of-section map ($x = \frac{1}{2}, p_x > 0$) in the plane (y, p_y). The three sections 2, 3 and 4 of domain 1 of diagram B represent three surface-of-section maps corresponding to three values of $g = 9, 1, 0.0625$, for fixed $h = 2$ ($M^4_{\Re} \sim T$). Section 1 represents bifurcation $T + 2C + \Re^2 \to C + \Re^2$: the fixed point in this figure corresponds to the periodic solution. $M^4_{\Re}$ is a circle $S(g = 25, h = 2)$ on the curve L_1 of diagram B.

We obtain

$$u(t) = u_3 - (u_3 - u_1)\,\mathrm{dn}^2\left(\frac{\sqrt{u_2 - u_1}}{2}t,\ \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}\right)$$

where dn is the Jacobi elliptic function.

In the plane (x, y), the Cartesian equation of the solution on the curve L_1 (parabolic mode) is

$$y^2 = \frac{c}{2(2h + c^2)}x^4 - \frac{c}{4}.$$

4. Numerical illustration

Using a surface-of-section map, we give a numerical illustration of the topological analysis studied in section 2.

For fixed values of energy h, as g varies the Liouville tori contained in the level set $H = h, G = g$ change their topological type. The surface-of-section map shown in figure 3 gives an illustration of the sequence of bifurcations of Liouville tori. This map is constructed using the method introduced by Poincaré and extended by Hénon [15].

5. Conclusion

The main concern of this paper is to study, through the Liouville tori and their bifurcations analytically and numerically, the phase space topology of the motion of particle subjected to velocity-dependent forces (Fokker–Planck). The kind of corresponding Hamiltonians associated with these systems are known to be not easily intrinsically, if not at all, separable. By means of a canonical transformation, we have been able to convert the previous system into the well known separable system of two coupled anisotropic quartic anharmonics.

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